

ON THE DECAY OF CROSSING NUMBERS OF SPARSE GRAPHS

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ABSTRACT. Richter and Thomassen proved that every graph has an edge e such that the crossing number $\text{cr}(G - e)$ of $G - e$ is at least $(2/5)\text{cr}(G) - O(1)$. Fox and Cs. Tóth proved that dense graphs have large sets of edges (proportional in the total number of edges) whose removal leaves a graph with crossing number proportional to the crossing number of the original graph; this result was later strengthened by Černý, Kynčl and G. Tóth. These results make our understanding of the decay of crossing numbers in dense graphs essentially complete. In this paper we prove a similar result for large sparse graphs in which the number of edges is not artificially inflated by operations such as edge subdivisions. We also discuss the connection between the decay of crossing numbers and expected crossing numbers, a concept recently introduced by Mohar and Tamon.

1. INTRODUCTION

The *crossing number* $\text{cr}(G)$ of a graph G is the minimum number of pairwise crossings of edges in a drawing of G in the plane. A graph G is *k-crossing-critical* if $\text{cr}(G) \geq k$, but $\text{cr}(G - e) < k$ for every edge e of G . Since loops are totally irrelevant for crossing number purposes, all graphs under consideration are loopless.

1.1. The decay of crossing numbers. In this paper we are concerned with the effect of edge removal in the crossing number of a graph (following Fox and Tóth [10], this is referred to as the *decay of crossing numbers*). Richter and Thomassen [22] proved that every graph G has some edge e such that $\text{cr}(G - e) \geq (2/5)\text{cr}(G) - 37/5$. They conjectured that there always exist an edge e such that $\text{cr}(G - e) \geq \text{cr}(G) - c\sqrt{\text{cr}(G)}$, for some universal constant c . This conjecture was proved by Fox and Tóth [10] for dense graphs.

Fox and Tóth actually proved a much stronger result: the existence of a large subset of edges whose removal leaves a graph whose crossing number

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is at least a proportion of the crossing number of the original graph. More precisely, they proved that for every fixed $\epsilon > 0$, there is a constant $n_0 = n_0(\epsilon)$ such that if G is a graph with $n > n_0$ vertices and $m > n^{1+\epsilon}$ edges, then G has a subgraph G' with at most $(1 - \frac{\epsilon}{24})m$ edges such that $cr(G') \geq (\frac{1}{28} - o(1))cr(G)$.

This result was further strengthened by Černý, Kynčl and G. Tóth [5], who proved that for every $\epsilon, \gamma > 0$ there is an $n_0 = n_0(\epsilon, \gamma)$ such that if G is a graph with $n > n_0$ vertices and $m > n^{1+\epsilon}$ edges, then G has a subgraph G' with at most $(1 - \frac{\epsilon\gamma}{1224})m$ edges such that $cr(G') \geq (1 - \gamma)cr(G)$.

1.2. The decay of crossing numbers of sparse graphs. Due to the Fox-Tóth and the Černý-Kynčl-Tóth results, our understanding of the decay of crossing numbers of dense graphs is essentially complete. The situation for sparse graphs is quite different. Although the Richter and Thomassen result is fully general, it only guarantees the existence of a single edge whose deletion leaves a graph with crossing number substantially large. As pointed out in [10], by combining the following two facts one obtains an improvement to the Richter-Thomassen result for graphs with n vertices and $m > 8.1n$ edges: (i) every graph with $m \geq \frac{103}{16}n$ satisfies $cr(G) \geq 0.032 \frac{m^3}{n^2}$ [20]; and (ii) for any graph G and any edge e of G , $cr(G - e) \geq cr(G) - m + 1$ [21].

In this paper we investigate the decay of crossing numbers of sparse graphs. We are particularly interested in establishing results as similar as possible as those in [10] and [5]: the existence of large sets of edges whose removal leaves a graph whose crossing number is at least some (constant) fraction of the crossing number of the original graph.

In contrast with dense graphs, in a sparse graph it is possible to artificially increase the number of edges of a graph, while maintaining its crossing number, without adding any substantial topological feature. Consider, for instance, a graph consisting of a large planar grid plus an additional edge e joining two vertices far apart; subdivide this additional edge r times (for some integer $r > 0$) to get a path P , and let G denote the resulting graph. For any given $\alpha > 0$, we can make r sufficiently large so that any set of at least $\alpha|E(G)|$ edges of G contains at least an edge of P . That is, for any set E_0 of at least $\alpha|E(G)|$ edges of G , the crossing number of $G - E_0$ is 0.

This example shows that no general result can possibly be established if we allow the number of edges to be artificially inflated. In particular, degree 2 vertices need to be precluded from the graphs under consideration. This is a particular instance of a more general way to spuriously increase the number of edges, by substituting a set of (possibly just one) edges joining the same two vertices by a plane connected graph, as we now describe.

We first recall the definition of a bridge. Let G be a graph, and let u, v be distinct vertices of G . Following Tutte, a uv -bridge is either a single edge joining u and v , together with u and v (in which case it is *trivial*), or a subgraph of G obtained by adding to a connected component K of $G \setminus \{u, v\}$

all the edges attaching K to u or v , together with their ends. A uv -bridge is uv -planar if it can be embedded in the plane with u and v in the same face.

Suppose that u, v are distinct vertices incident with the same face in a connected plane graph H with $|V(H)| > 2$, and let k be the maximum number of pairwise edge-disjoint uv -paths in H . We say that (H, u, v) is a uv -blob of width k . Now consider a graph G , and let u, v be vertices of G , joined by $k \geq 1$ edges. It is easy to see that we may substitute the edges joining u and v by an arbitrarily large uv -blob of width k , leaving the crossing number (and the criticality of G , if G is critical) unchanged. Conversely, if G is a graph with a vertex cut $\{u, v\}$, and for some $\{u, v\}$ -bridge H we have that (H, u, v) is a uv -blob of width k , then G may be simplified, leaving its crossing number (and its criticality, if G is critical) unchanged, by substituting H by k parallel uv -edges.

Note that the concept of uv -blob captures, in particular, the operation of edge subdivision. Indeed, a subdivided edge is simply a uv -blob of width 1, all of whose vertices, other than u and v , have degree 2.

1.3. The main result. Since we are interested in proving the existence of large sets of edges (linear in the crossing number) with a special property (their removal does not decrease the crossing number arbitrarily), we need to preclude the existence of $\{u, v\}$ -bridges (for any pair u, v of vertices) that are uv -blobs, since they inflate the number of edges of a graph, while adding no topologically interesting structure whatsoever to the graph itself.

As it happens, such objects are the *only* structure that needs to be avoided. A graph is *irreducible* if there do not exist vertices u, v and a $\{u, v\}$ -bridge H such that (H, u, v) is a uv -blob. We prove that if G is irreducible, then a large set of its edges (linear in the crossing number) may be removed, and still leave a graph whose crossing number is at least a fraction of the crossing number of the original graph. More precisely:

Theorem 1. *For each $\epsilon > 0$ and each positive integer k there exist $m_0 := m_0(\epsilon, k)$ and $\gamma := \gamma(\epsilon)$ with the following property. Every 2-connected irreducible graph G with $\text{cr}(G) = k$ and at least m_0 edges has a set E_0 of at least γk edges such that $\text{cr}(G - E_0) > (1/2 - \epsilon)\text{cr}(G)$.*

Trivially, 3-connected graphs are irreducible, so in particular Theorem 1 applies to all 3-connected graphs.

We also apply our techniques to improve (for sufficiently large graphs) the Richter and Thomassen result on crossing-critical graphs. Richter and Thomassen proved in [22] that every graph G has an edge e such that $\text{cr}(G - e) \geq (2/5)\text{cr}(G) - 37/5$.

In order to improve on this result, again we need to be careful not to allow the artificial inflation in the number of edges. However, we do not need the full condition of irreducibility: it suffices to require that each vertex is adjacent to at least 3 other vertices. A slight variant of this requirement (namely *X-minimality*) was introduced by Ding, Oporowski, Thomas, and

106 Vertigan in [6], with the same motivation of not allowing a graph with given
 107 crossing number (in their case, a 2-crossing-critical graph) to spuriously grow
 108 its number of edges.

109 **Theorem 2.** *For each positive integer k , there is an integer $m_1 := m_1(k)$
 110 with the following property. Let G be a 2-connected graph in which each
 111 vertex is adjacent to at least 3 vertices. If $\text{cr}(G) = k$ and G has at least m_1
 112 edges, then G has an edge e such that $\text{cr}(G - e) > (2/3)\text{cr}(G) - 10^8$.*

113 We conclude this section with a brief overview of the proofs of Theorems 1
 114 and 2, and of the rest of this paper.

115 As in [5], [10], and [22], we make essential use of the embedding method.
 116 This technique consists of finding a set E_0 of edges in a graph G , and for
 117 each $e = uv \in E_0$ a set of pairwise edge-disjoint uv -paths $\mathcal{P}(e)$, with the
 118 aim of drawing $G - E_0$ (with $\text{cr}(G - E_0)$ crossings) and then embedding each
 119 $e \in E_0$ very closely to some path in $\mathcal{P}(e)$. The idea is to choose the set E_0
 120 so that the embedding can be done without adding too many crossings.

121 Richter and Thomassen proved the existence of an edge $e = uv$ (so that
 122 $E_0 = \{e\}$) with the property that there is a uv -path (that avoids e) of length
 123 at most 4, all of whose internal vertices have degree less than 12. Fox and
 124 Tóth, and Černý-Kynčl-Tóth used the density of G to show the existence
 125 of a large set E_0 of edges, such that each edge $e = uv$ of E_0 has a large
 126 collection $\mathcal{P}(e)$ of short edge-disjoint paths, and such that the collections
 127 $\mathcal{P}(e)$ are pairwise edge-disjoint.

128 In our current setup (sparse graphs) for all we know the graphs under
 129 consideration may have maximum degree 3, and so in general we cannot
 130 expect to find collections $\mathcal{P}(e)$ of more than two edge-disjoint paths, for
 131 each $e \in E_0$. We prove that, indeed, each graph under consideration has
 132 large set E_0 of edges such that each $e = uv \in E_0$ has two short uv -paths
 133 $P(e), Q(e)$ whose internal vertices have bounded degree, and if $e \neq f$ then
 134 $P(e) \cup Q(e)$ and $P(f) \cup Q(f)$ are edge-disjoint. As it happens, $P(e)$ and $Q(e)$
 135 are not necessarily edge-disjoint, but this turns out to be unimportant. To
 136 be slightly more precise, let us mention that each graph $\Xi = e \cup P(e) \cup Q(e)$
 137 has the property that $P(e)$ and $Q(e)$ have length at most ℓ , and the degree
 138 of their internal vertices is less than Δ . Following the lively notation in [5],
 139 we call each Ξ an (ℓ, Δ) -earring.

140 Most of the rest of this paper is devoted to proving the result described
 141 in the previous paragraph. We start by establishing, in Section 2, several
 142 assorted statements on planar graphs; these are, in one way or another,
 143 elementary consequences of Euler's formula. The existence of a large set
 144 of edge-disjoint (ℓ, Δ) -earrings (for certain values of ℓ and Δ) is proved
 145 in Section 3 for planar graphs, and in Section 4 for irreducible nonplanar
 146 graphs.

147 In Section 5 we establish the version of the embedding method that we
 148 need. The proofs of Theorems 1 and 2 are in Section 6.

149 In Section 7 we discuss the connection between the decay of crossing
 150 numbers and the concept, recently introduced by Mohar and Tamon [18], of
 151 expected crossing numbers. Finally, in Section 8 we present some concluding
 152 remarks and open questions.

153 2. ASSORTED LEMMAS ON PLANAR GRAPHS

154 A *branch* in a graph is a path whose endpoints have degree at least 3, and
 155 all whose internal vertices have degree 2.

156 **Lemma 3.** *Let $G = (V, E)$ be a planar graph with minimum degree at least*
 157 *2, and let $B \subseteq V$ be a set of vertices of degree at least 3. Suppose that the*
 158 *number of branches with both endpoints in B is at most s . Then there are at*
 159 *least $|V|/2 - s/2 - (3/2)|B|$ edges with both endpoints in $V \setminus B$.*

160 *Proof.* Let $W := V \setminus B$. To help comprehension, we color white (respectively,
 161 black) the vertices in W (respectively, B). A branch is *black* if its endpoints
 162 are both black. A white vertex is *black-covered* if all its adjacent vertices are
 163 black. A black-covered vertex is *of Type I* if it has degree 2; otherwise (that
 164 is, if it has degree ≥ 3) it is *of Type II*.

165 Since there are no black vertices of degree 2, then no black branch can
 166 contain more than one Type I vertex. Thus there are at most s Type I
 167 vertices.

168 Let W' denote the set of black-covered vertices of Type II, and let G'
 169 denote the subgraph of G induced by the edges incident with a vertex in
 170 W' . This is a bipartite graph with bipartition (W', B') , for some $B' \subseteq B$.
 171 A standard Euler formula argument yields that $|E(G')| \leq 2|V(G')| - 4 =$
 172 $2|W'| + 2|B'| - 4$. Since each vertex in W' has degree at least 3 (in G' , as
 173 well as in G) it follows that $|E(G')| = \sum_{v \in W'} d(v) \geq 3|W'|$. Thus $3|W'| \leq$
 174 $2|W'| + 2|B'| - 4 \leq 2|W'| + 2|B| - 4$, and so $|W'| \leq 2|B| - 4$. Thus, there
 175 are at most $2|B| - 4$ Type II vertices.

176 Therefore, the total number of black-covered vertices is at most $s + 2|B| - 4$.
 177 It follows that there are at least $|W| - s - 2|B| + 4 > |W| - s - 2|B|$ white
 178 vertices that are adjacent to at least one white vertex, and so there are at
 179 least $|W|/2 - s/2 - |B| = |V|/2 - s/2 - (3/2)|B|$ edges with both endpoints
 180 in W . \square

181 The *length* of a face in a plane graph is the length of its boundary walk.

182 A *digon* in an embedded graph consists of two parallel edges, together
 183 with their common endpoints. If the endpoints are u and v , then it is a *uv-*
 184 *digon*. A plane embedding of a graph G is *clean* if for each pair of vertices
 185 u, v joined by parallel edges, there exist edges e, e' with endpoints u and v ,
 186 such that the disc bounded by the digon formed by e and e' contains all
 187 edges parallel to e and e' , and no other edges.

188 **Lemma 4.** *Let G be a connected plane graph in which each vertex is adjacent*
 189 *to at least 3 vertices. Suppose that the embedding of G is clean. Let $r \geq 0$*

190 be an integer. Let F be the set of faces of G , and let F' be the set of those
 191 faces whose length is at most $r + 5$. Then $|F'| \geq \frac{r|F|+12}{r+3}$.

192 *Proof.* Let H be a graph obtained from G as follows: for each pair (u, v) of
 193 vertices joined by parallel edges, contract to a single all the parallel edges
 194 between u and v . Let F_H denote the set of faces of H , and let F'_H denote
 195 the set of faces of H with length at most $r + 5$. Our first task is to show
 196 that $|F'_H| \geq \frac{r|F_H|+12}{r+3}$.

197 For each $f \in F_H$ the sum $w(f) := \sum_{v \sim f} 1/d(v)$ is the *weight* of f , where
 198 $d(v)$ denotes the degree of the vertex v and $v \sim f$ means that v is incident
 199 with f . (A vertex v contributes to $w(f)$ as many times as the boundary
 200 walk of f passes through v .) Since H is simple and has minimum degree
 201 at least 3, then, letting $l(f)$ denote the length of f , we have $l(f) \geq 3$ and
 202 $w(f) \leq l(f)/3$. It is easy to see that $|V(H)| = \sum_{f \in F_H} w(f)$ and $2|E(H)| =$
 203 $\sum_{f \in F_H} l(f)$. From the last two equations and Euler's formula it follows that
 204 $2 = \frac{1}{2} \sum_{f \in F_H} \{2w(f) - l(f) + 2\}$.

205 Since $w(f) \leq l(f)/3$, we have

$$12 \leq \sum_{f \in F_H} \{-l(f) + 6\} = \sum_{f \in F'_H} \{-l(f) + 6\} + \sum_{f \in F_H - F'_H} \{-l(f) + 6\}.$$

206 Since $l(f) \geq 3$ for each $f \in F_H$, then $-l(f) + 6 \leq 3$ and thus $\sum_{f \in F'_H} \{-l(f) +$
 207 $6\} \leq 3|F'_H|$. If $f \in F_H - F'_H$ then $l(f) - 6 \geq r$, that is, $-l(f) + 6 \leq -r$, and so
 208 $\sum_{f \in F_H - F'_H} \{-l(f) + 6\} \leq -r(|F_H| - |F'_H|)$. Thus, $12 \leq 3|F'_H| - r(|F_H| - |F'_H|)$,
 209 and so $|F'_H| \geq \frac{r|F_H|+12}{r+3}$, as required.

210 Now as we inflate back H to G , each face in F'_H becomes a face in F' .
 211 The other faces in F' are precisely the $t := |E(G) \setminus E(H)|$ faces created
 212 in the inflation process, that is, those bounded by parallel edges. Thus
 213 $|F| = |F_H| + t$ and $|F'| = |F'_H| + t$. Thus $|F'| - t \geq \frac{r(|F| - t) + 12}{r + 3}$, and so
 214 $|F'| \geq \frac{r|F| + 12}{r + 3} + t(1 - \frac{r}{r + 3}) \geq \frac{r|F| + 12}{r + 3}$. \square

215 If D is a digon in a plane graph, then the open (respectively, closed) disc
 216 bounded by D will be denoted $\Delta(D)$ (respectively, $\overline{\Delta}(D)$). If D, D' are
 217 digons, then we write $D' \leq D$ if $\Delta(D') \subseteq \Delta(D)$. We recall that a vertex of
 218 degree 0 is an *isolated vertex*.

219 **Proposition 5.** *Let $G = (V, E)$ be a plane graph, and let Z be a set of*
 220 *isolated vertices of G . Suppose that for each digon D in G , the disc bounded*
 221 *by D contains at least one vertex in Z . Then G has at most $3|V \setminus Z| + |Z|$*
 222 *edges.*

223 *Proof.* Let $Y := V \setminus Z$. To help comprehension, we colour the vertices in Y
 224 and Z black and green, respectively.

225 We prove the stronger statement that G has at most $3|Y| + |Z| - 6$ edges.
 226 We proceed by induction on the number of digons in G . In the base case G

has no digons, and so by Euler's Formula it has at most $3|Y| - 6$ edges, as required. For the inductive step, we assume that G has at least one digon, and let D be a \leq -minimal digon in G .

Suppose first that D is also \leq -maximal. Then let G' be the graph obtained from G by removing one edge of D and one green vertex contained in $\Delta(D)$. Now G' contains one fewer edge and one fewer green vertex than G . It is easy to see that the induction hypothesis can be applied to G' , and so the inductive step follows.

Therefore we may assume that D is not \leq -maximal. Among all digons that contain D , let D' be a \leq -minimal one.

Suppose that D and D' have an edge e in common, and let \bar{e} be the other edge of D . It is easy to see that the induction hypothesis can be applied to the graph obtained from G by removing \bar{e} and a green vertex contained in $\Delta(D)$, and once again the inductive step follows. Thus we may assume that D and D' do not have an edge in common.

If $\Delta(D')$ contains a green vertex not contained in $\Delta(D)$, the situation is again straightforward: the induction hypothesis can be applied to the graph G' obtained by removing one edge of D and one green vertex contained in $\Delta(D)$, and the inductive step follows. Thus we may assume that every green vertex contained in $\Delta(D')$ is contained in $\Delta(D)$.

In this case, there are no digons other than D' and D contained in $\bar{\Delta}(D')$. Now let G' be the graph obtained by removing from G the black vertices and all the edges contained in $\Delta(D')$. Let Y' and Z' denote the sets of black and green vertices of G' , respectively, and let E' denote the set of edges of G' (note that $Z' = Z$). We may clearly apply the induction hypothesis to G' , obtaining that $|E'| \leq 3|Y'| + |Z| - 6$. Let $Y'' := Y \setminus Y'$, and $E'' := E \setminus E'$. Let x, y be the vertices of D' . Consider the graph G'' that consists of the vertices in $Y'' \cup \{x, y\}$ and the edges in E'' . Since G'' has exactly one digon (namely D), the usual Euler formula argument yields $|E(G'')| \leq 3|V(G'')| - 5$. However, this inequality is tight only if G'' is maximally planar, that is, if no edge can be added between two nonadjacent vertices while maintaining planarity; thus, since x and y are not adjacent in G'' , it follows that $|E(G'')| \leq 3|V(G'')| - 6$. Thus $|E''| \leq 3(|Y''| + 2) - 6$. That is, $|E| - |E'| \leq 3(|Y| - |Y'| + 2) - 6$, and so $|E| \leq 3|Y| + |Z| - 6$, as required. \square

A set Z of vertices in a 2-connected planar graph G is an *anchor* if the following hold:

- (1) no vertex in Z is part of a 2-vertex-cut in G ; and
- (2) if $\{u, v\}$ is a 2-vertex-cut in G , then every nontrivial uv -bridge contains a vertex in Z .

Lemma 6. *Let G be a 2-connected plane graph in which each vertex is adjacent to at least 3 distinct vertices, and let Z be an anchor of G . Let $Y \subseteq V(G) \setminus Z$, and let E_Y denote the set of edges of G with both endpoints in Y . Then the number of faces of G that are incident with exactly 2 vertices of Y is at most $3|Y| + |Z| + |E_Y|$.*

271 *Proof.* We may assume that $|Y| \geq 2$, as otherwise there is nothing to prove.
 272 Let F_2 denote the set of faces of G that are incident with exactly two vertices
 273 of Y .

274 We start by coloring red each edge in E_Y , and green each vertex in Z .
 275 Now for each $f \in F_2$, join the two vertices in Y incident with f by a simple
 276 blue arc contained (except, obviously, for its endpoints) in f . Let H denote
 277 the plane graph that consists of the vertices in Y plus all the red edges and
 278 the blue arcs (now seen as edges), as well as the set Z of green vertices.
 279 Note that the green vertices are isolated in H . We remark that $|F_2|$ is the
 280 number of blue edges in H .

281 Note that if D is a blue digon in H (that is, both edges of D are blue),
 282 with vertices u and v , then $\overline{\Delta}(D)$ contains a uv -bridge in G . This bridge
 283 may be trivial (in which case it is a red edge) or nontrivial (in which case,
 284 by hypothesis, $\Delta^o(D)$ contains a green vertex).

285 Finally, let K denote the graph that results from H by substituting each
 286 red edge by an isolated red vertex (placed in the interior of the red edge).
 287 Note that $|E(K)| = |F_2|$, that the vertex set of K is the union of Y with
 288 the set of all green or red vertices, and that there are $|Z|$ green and $|E_Y|$
 289 red vertices.

290 The graph K has the property that for each (necessarily blue) digon D in
 291 K , $\Delta^o(D)$ contains either a green or a red vertex. Applying Proposition 5
 292 we obtain that $|E(K)| \leq 3|Y| + |Z| + |E_Y|$. Thus $|F_2| \leq 3|Y| + |Z| + |E_Y|$,
 293 as required. \square

294 If G is a plane graph, then we let G^o denote its dual.

295 **Lemma 7.** *Let G be a 2-connected plane graph, and let Z be an anchor of*
 296 *G . Suppose that the embedding of G is clean. Let F' be a set of faces of G*
 297 *of length at least 3. Then the number of branches in G^o with both endpoints*
 298 *in F' is at most $3|F'| + |Z|$.*

299 *Proof.* Since the embedding is clean, we may as well assume (in the context
 300 of this lemma) that G has no parallel edges. It follows that all branches with
 301 both endpoints in F' are actual edges in G^o . Thus our goal is to show that
 302 there are at most $3|F'| + |Z|$ edges in G^o with both endpoints in F' .

303 Regarding G and G^o as simultaneously embedded, remove everything
 304 except for F' (seen as a set of vertices in G^o), the edges (in G^o) joining
 305 two vertices in F' , and the vertices in Z . The result is a graph G' in which
 306 each vertex in Z is isolated, and such that the disc bounded by every digon
 307 contains a vertex in Z . To see this last property, note that if e and f are
 308 the edges of a digon in G' , then the edges in G corresponding to e and f
 309 are a 2-edge-cut in G ; since Z is an anchor set of G , it then follows that the
 310 disc bounded by the digon must contain a vertex of Z in its interior.

311 Applying Proposition 5, we obtain that G' has at most $3|F'| + |Z|$ edges.
 312 This finishes the proof, since there is a bijection between the edges in G'
 313 and the edges in G^o with both endpoints in F' . \square

3. EARRINGS IN PLANAR GRAPHS

314

315 Černý, Kynčl and Tóth introduced the lively terminology of *earring of*
 316 *size* p to describe a graph consisting of an edge $e = uv$ plus a collection of
 317 p pairwise edge-disjoint, bounded-length uv -paths. In order to use the re-
 318 embedding method, the goal is to find many pairwise edge-disjoint earrings.

319 As we mentioned in Section 1, in our current context of sparse graphs,
 320 where (for all we know) the graphs under consideration may have maximum
 321 degree 3, the best we could hope for is to prove the existence of a large
 322 collection of earrings, each of size 2. As we also mentioned, in this discussion
 323 we do not need the two uv -paths of each earring to be edge-disjoint, but only
 324 a weaker condition (see (iii) in the following definition).

325 Let ℓ, Δ be positive integers. An (ℓ, Δ) -*earring* of a graph G is a subgraph
 326 of G that consists of a *base* edge $e = uv$ plus two distinct uv -paths P, Q
 327 (disjoint from e) with the following properties: (i) each of P and Q has at
 328 most ℓ edges; (ii) each internal vertex of P or Q has degree less than Δ ; and
 329 (iii) if f is an edge in both P and Q , then $\{e, f\}$ is a 2-edge-cut of G .

330 An edge $e = uv$ in a 2-connected plane graph is an (ℓ, Δ) -*edge* if each of
 331 its two incident faces has length at most $\ell + 1$, and no vertex incident with
 332 these two faces, other than possibly u or v , has degree Δ or greater. If e is an
 333 (ℓ, Δ) -edge, then the subgraph that consists of e plus the cycles that bound
 334 its two incident faces, is an (ℓ, Δ) -earring, the (ℓ, Δ) -earring $\Xi(e)$ *associated*
 335 *to* e .

336 The following lemma is the main workhorse in this paper.

337 **Lemma 8.** *Let $G = (V, E)$ be a 2-connected planar graph in which each*
 338 *vertex is adjacent to at least 3 other vertices. Let Z be an anchor of G ,*
 339 *where each vertex in Z has degree 4. Then G has at least $10^{-10}|E| - 10^{-5}|Z|$*
 340 *pairwise edge-disjoint $(5000, 500)$ -earrings.*

341 *Proof.* Throughout the proof, we make use of several constants that are
 342 either very small, very close to 1, or somewhat large. In order to simplify
 343 the whole discussion, we first proceed to introduce these constants. We let
 344 $\ell_0 = 5000$, $\Delta_0 = 500$, $\mathbf{c}_1 = 10^{-10}$, $\mathbf{c}_2 = 10^{-5}$, $\mathbf{c}_3 = 999/1000$, $\mathbf{c}_4 = 1/1000$,
 345 $\mathbf{c}_5 = 999$, $\mathbf{c}_6 = 36/5000$, and $\mathbf{z}_1 = 3(10^{-10})$.

346 It is a trivial observation that every planar graph has a clean plane em-
 347 bedding (clean embeddings are defined before Lemma 4). Throughout the
 348 proof we consider a fixed clean embedding of G in the plane. Let F denote
 349 the set of all faces of G , and let $t := |Z|$.

350 **Claim 9.** *It suffices to show that there are at least $(2\ell_0(2\ell_0 + 1) + 1) \cdot (\mathbf{z}_1|F| -$*
 351 *$\mathbf{c}_2 t)$ (ℓ_0, Δ_0) -edges.*

352 *Proof.* Consider the graph H whose vertices are the (ℓ_0, Δ_0) -edges of G ,
 353 with two distinct (ℓ_0, Δ_0) -edges e, f adjacent if $\Xi(e)$ and $\Xi(f)$ have some
 354 edge in common.

355 We note that H has maximum degree at most $2\ell_0(2\ell_0 + 1)$. This follows
 356 at once from the following two easy observations: (i) for each (ℓ_0, Δ_0) -edge

357 e , $\Xi(e)$ has at most $2\ell_0$ edges other than e ; and (ii) each edge of G belongs
 358 to at most $2\ell_0 + 1$ (ℓ_0, Δ_0) -earrings of the form $\Xi(f)$ for some edge f .

359 Thus, $V(H)$ has a stable set of size at least $|V(H)|/(2\ell_0(2\ell_0 + 1) + 1)$.
 360 Suppose that G has at least $(2\ell_0(2\ell_0 + 1) + 1) \cdot (\mathbf{z}_1|F| - \mathbf{c}_2t)$ (ℓ_0, Δ_0) -edges;
 361 that is, $|V(H)| \geq (2\ell_0(2\ell_0 + 1) + 1) \cdot (\mathbf{z}_1|F| - \mathbf{c}_2t)$. Then H has a stable set S
 362 of size at least $\mathbf{z}_1|F| - \mathbf{c}_2t$; that is, there is a collection of at least $\mathbf{z}_1|F| - \mathbf{c}_2t$
 363 pairwise edge-disjoint (ℓ_0, Δ_0) -earrings.

364 Since G has minimum degree at least 3, a routine Euler formula argument
 365 yields that $|F| \geq |E|/3 + 2$. Thus there are at least $\mathbf{z}_1(|E|/3 + 2) - \mathbf{c}_2t >$
 366 $\mathbf{c}_1|E| - \mathbf{c}_2t$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings, as required in Lemma 8.
 367 \square

368 Let W be the set of those vertices of G with degree at least Δ_0 , and let
 369 F_W denote the set of faces of G that are incident with some vertex in W .
 370 For each integer $j \geq 1$, let F_j denote the set of those faces of G incident
 371 with exactly j vertices in W (and perhaps other vertices in $V \setminus W$), and let
 372 $f_j = |F_j|$. Note that F_W is the disjoint union $\bigcup_{i \geq 1} F_i$.

373 Let F_{long} (respectively, F_{short}) denote the collection of faces of G with
 374 length greater than (respectively, at most) $\ell_0 + 1$, and let $f_{\text{long}} := |F_{\text{long}}|$ and
 375 $f_{\text{short}} := |F_{\text{short}}|$. It follows immediately from Lemma 4 that

$$(1) \quad f_{\text{short}} \geq \mathbf{c}_3|F|.$$

376 Since F is the disjoint union of F_{long} and F_{short} , then $|F| = f_{\text{long}} + f_{\text{short}}$,
 377 and so $f_{\text{short}} \geq \mathbf{c}_3(f_{\text{long}} + f_{\text{short}})$ implies $f_{\text{short}} \geq (\mathbf{c}_3/(1 - \mathbf{c}_3))f_{\text{long}}$. Note that
 378 $\mathbf{c}_5 = \mathbf{c}_3/(1 - \mathbf{c}_3)$. Therefore,

$$(2) \quad f_{\text{short}} \geq \mathbf{c}_5 f_{\text{long}}.$$

379 We note that $\sum_{u \in W} d(u) = \sum_{i \geq 1} i f_i$. A routine application of Euler's
 380 formula yields that $\sum_{i \geq 3} i f_i \leq 2(3|W| - 6) = 6|W| - 12$. Since all vertices of
 381 Z have degree 4 it follows that $W \subseteq V \setminus Z$, and so we can apply Lemma 6,
 382 to obtain $f_2 \leq 3|W| + t + |E_W|$. Combining these observations we obtain

$$(3) \quad f_1 \geq \sum_{u \in W} d(u) - 12|W| - 2|E_W| - 2t + 12.$$

383 **Claim 10.** *If $|F_W| > 24t + 24\mathbf{c}_4 f_{\text{short}}$, then Lemma 8 follows.*

384 *Proof.* We establish four subclaims, and finally show that the proof follows
 385 easily from them.

386 **SUBCLAIM A** *If $|E_W| > 6|W| - 12 + \mathbf{c}_4 f_{\text{short}}$, then Lemma 8 follows.*

387 *Proof.* If e_1, e_2, e_3 are parallel edges with common endpoints u, v , and e_2 is
 388 in the disc bounded by the digon formed by e_1 and e_3 , then e_2 is a *sheltered*
 389 edge. By Euler's formula, a simple graph on $|W|$ vertices has at most $3|W| - 6$
 390 edges. Since the embedding of G is clean, it follows that the subgraph of G
 391 induced by W has at least $|E_W| - 2(3|W| - 6) = |E_W| - 6|W| + 12$ sheltered

edges. The fact that G is clean also implies that each sheltered edge is a (ℓ_0, Δ_0) -edge, and so G has at least $|E_W| - 6|W| + 12$ (ℓ_0, Δ_0) -edges.

Suppose that $|E_W| > 6|W| - 12 + \mathbf{c}_4 f_{\text{short}}$. Then G has at least $\mathbf{c}_4 f_{\text{short}}$ (ℓ_0, Δ_0) -edges. Using (1), it follows that G has at least $\mathbf{c}_3 \mathbf{c}_4 |F|$ (ℓ_0, Δ_0) -edges. The result now follows from Claim 9, since $\mathbf{c}_3 \mathbf{c}_4 > (2\ell_0(2\ell_0 + 1) + 1)\mathbf{z}_1$. \square

SUBCLAIM B *If $(1/6)(\sum_{u \in W} d(u)) \leq 12|W| + 2t + 2|E_W| - 12$, then $|F_W| < 24t + 24\mathbf{c}_4 f_{\text{short}}$ or else Lemma 8 follows.*

Proof. By Subclaim A, under the given hypothesis we may assume that $\sum_{u \in W} d(u) \leq 72|W| + 12t + (72|W| - 144 + 12\mathbf{c}_4 f_{\text{short}}) - 72 = 144|W| + 12t + 12\mathbf{c}_4 f_{\text{short}} - 216 < 144|W| + 12t + 12\mathbf{c}_4 f_{\text{short}}$.

Since each vertex in W has degree at least Δ_0 , it follows that $\Delta_0|W| \leq \sum_{u \in W} d(u)$. Hence, $|W| < (12t + 12\mathbf{c}_4 f_{\text{short}})/(\Delta_0 - 144)$. On the other hand, obviously $|F_W| \leq \sum_{u \in W} d(u)$, and so $|F_W| < 144(12t + 12\mathbf{c}_4 f_{\text{short}})/(\Delta_0 - 144) + 12t + 12\mathbf{c}_4 f_{\text{short}}$. Since $144/(\Delta_0 - 144) \leq 1$, this implies $|F_W| < 12t + 12\mathbf{c}_4 f_{\text{short}} + 12t + 12\mathbf{c}_4 f_{\text{short}} = 24t + 24\mathbf{c}_4 f_{\text{short}}$. \square

SUBCLAIM C *If $(1/6)(\sum_{u \in W} d(u)) \leq f_{\text{long}}$, then $|F_W| \leq 6f_{\text{short}}/\mathbf{c}_5$.*

Proof. Suppose that $(1/6)(\sum_{u \in W} d(u)) \leq f_{\text{long}}$. The obvious inequality $|F_W| \leq \sum_{u \in W} d(u)$ then implies that $|F_W| \leq 6 \cdot f_{\text{long}}$. The required inequality follows from (2). \square

SUBCLAIM D *If $(1/6)(\sum_{u \in W} d(u)) > 12|W| + 2t + 2|E_W| - 12$ and $(1/6)(\sum_{u \in W} d(u)) > f_{\text{long}}$, then $|F_W| \leq \mathbf{c}_6 f_{\text{short}}$ or else Lemma 8 follows.*

Proof. We show that, under the given hypotheses, if $|F_W| > \mathbf{c}_6 f_{\text{short}}$, then there are at least $(\mathbf{c}_3 \mathbf{c}_6/3)|F|$ (ℓ_0, Δ_0) -edges; the subclaim then follows from Claim 9, since $(\mathbf{c}_3 \mathbf{c}_6)/3 \geq (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{z}_1$.

It follows that, under the current hypotheses,

$$(4) \quad f_{\text{long}} < (1/3) \sum_{u \in W} d(u) - 12|W| - 2t - 2|E_W| + 12.$$

Since $|F_1 \setminus F_{\text{long}}| \geq f_1 - f_{\text{long}}$, using (3) and (4) we obtain

$$|F_1 \setminus F_{\text{long}}| \geq \sum_{u \in W} d(u) - 12|W| - 2|E_W| - 2t + 12 - f_{\text{long}} > (2/3) \sum_{u \in W} d(u).$$

Since each face in F_1 is (by definition) incident with exactly one vertex in W , the inequality $|F_1 \setminus F_{\text{long}}| > (2/3) \sum_{u \in W} d(u)$ implies that at least $1/3$ of the edges incident with W have their two incident faces in $F_1 \setminus F_{\text{long}}$. Note that all such edges are (ℓ_0, Δ_0) -edges. We conclude that there are at least $(1/3) \sum_{u \in W} d(u)$ (ℓ_0, Δ_0) -edges incident with W . Since obviously $\sum_{u \in W} d(u) \geq |F_W|$, this implies that there are at least $|F_W|/3$ (ℓ_0, Δ_0) -edges.

424 Using the assumption $|F_W| > \mathbf{c}_6 f_{\text{short}}$ and (1), it follows that there are at
 425 least $(\mathbf{c}_3 \mathbf{c}_6 / 3) |F|$ (ℓ_0, Δ_0) -edges, as required. \square

426 We now complete the proof of Claim 10.

427 Since the hypotheses of Subclaims B, C, and D are exhaustive, it fol-
 428 lows from these subclaims that either we may assume that $|F_W| < 24t +$
 429 $24\mathbf{c}_4 f_{\text{short}}$, or $|F_W| \leq 6f_{\text{short}}/\mathbf{c}_5$, or we may assume that $|F_W| \leq \mathbf{c}_6 f_{\text{short}}$.
 430 Since $\max\{24\mathbf{c}_4, 6/\mathbf{c}_5, \mathbf{c}_6\} = 24\mathbf{c}_4$, it follows that we may assume that $|F_W| <$
 431 $24t + 24\mathbf{c}_4 f_{\text{short}}$. \square

432 We now complete the proof of Lemma 8.

433 A face is *white* if it is either in $F_{\text{short}} \setminus F_W$ or has length exactly 2, and
 434 is *black* otherwise. We let F_\circ (respectively, F_\bullet) denote the set of all white
 435 (respectively, black) faces. Let $f_\circ := |F_\circ|$, and $f_\bullet := |F_\bullet|$.

436 Now consider the dual G° of G . The 2-connectivity of G implies that G°
 437 is also 2-connected. Let us say that an edge in G° is *white* if its endpoints
 438 are both white (faces in G).

439 The key (and completely straightforward) observation is that the edge of
 440 G associated to each white edge is an (ℓ_0, Δ_0) -edge. Our final goal is to
 441 prove that there are many white edges.

442 Every face in F_\bullet is either in F_{long} or in F_W , and so $f_\bullet \leq f_{\text{long}} + |F_W|$.
 443 Using (2), Claim 10, and the obvious inequality $f_{\text{short}} \leq |F|$, we obtain

$$(5) \quad f_\bullet \leq 24t + (24\mathbf{c}_4 + 1/\mathbf{c}_5)|F|.$$

444 By Lemma 7, G° has at most $3f_\bullet + t$ branches with both endpoints black.
 445 Lemma 3 (applied to G°) then implies that there are at least $|F|/2 - (3f_\bullet +$
 446 $t)/2 - (3/2)f_\bullet = |F|/2 - 3f_\bullet - t/2 \geq (1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5))|F| - (145/2)t$
 447 white edges.

448 As we have observed, the edge of G associated to each white edge is an
 449 (ℓ_0, Δ_0) -edge. Thus there are at least $(1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5))|F| - (145/2)t$
 450 (ℓ_0, Δ_0) -edges. Since $1/2 - 3(24\mathbf{c}_4 + 1/\mathbf{c}_5) \geq (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{z}_1$ and
 451 $145/2 \leq (2\ell_0(2\ell_0 + 1) + 1) \cdot \mathbf{c}_2$, then we are done by Claim 9. \square

452 4. EARRINGS IN NONPLANAR GRAPHS

453 **Lemma 11.** *Let $G = (V, E)$ be a 2-connected irreducible graph. Then G*
 454 *has at least $10^{-10}|E| - (10^{-5} + 2)\text{cr}(G)$ pairwise edge-disjoint $(5000, 500)$ -*
 455 *earrings.*

456 *Proof.* Let $\ell_0 := 5000$, $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, $\mathbf{c}_2 := 10^{-5}$, and $\mathbf{c}_7 :=$
 457 $(10^{-5} + 2)$. Let $t := \text{cr}(G)$, and let \mathcal{D} be a drawing of G with exactly t
 458 crossings. Let H denote the plane graph that results by regarding the t
 459 crossings as degree 4 vertices (this is the *crossings-to-vertices conversion*),
 460 which we colour green to help comprehension (the other vertices of H , each
 461 of which corresponds to a vertex in G , are coloured black). We claim that
 462 (i) each vertex in H is adjacent to at least 3 other vertices; (ii) no green

vertex is part of a 2-vertex-cut; (iii) H is 2-connected; and (iv) the set of green vertices is an anchor set for H .

We start by noting that (i) follows easily from the irreducibility of G , plus the observation that in any crossing-minimal drawing of any graph, the two edges involved in any crossing cannot have a common endpoint.

By way of contradiction, suppose that u, v are green vertices such that $\{u, v\}$ is a 2-vertex-cut in H . It is easy to see that then there are exactly two uv -bridges. Let B be any of these uv -bridges, and let H' denote the plane graph obtained from H by performing a Whitney switching on B around u and v . Now by reversing the crossings-to-vertices conversion, we obtain from H' a drawing of G in which the edge intersections corresponding to u and v are tangential, not crossings. Each of these two tangential edge intersections may be removed with a small perturbation, yielding a drawing of G with two fewer crossings than \mathcal{D} , contradicting the crossing-minimality of \mathcal{D} . This contradiction shows that $\{u, v\}$ cannot be a 2-vertex-cut in H . A similar contradiction is obtained from the assumption that H has a 2-vertex-cut with exactly one green vertex (in this case one obtains a drawing of G with one fewer crossing than \mathcal{D}). This proves (ii).

The 2-connectedness of G readily implies that no black vertex can be a cut vertex of H . On the other hand, a similar switching argument as in the proof of (ii) shows that no green vertex can be a cut vertex of H . This proves (iii).

Now let u, v be black vertices such that $\{u, v\}$ is a 2-vertex-cut in H , and let B be a nontrivial uv -bridge. If B does not contain any green vertex, then (B, u, v) is clearly a uv -blob of G . Since this contradicts the irreducibility of G , (iv) follows.

We can thus apply Lemma 8 to H , and obtain that H has a collection \mathcal{E} of at least $\mathbf{c}_1|E(H)| - \mathbf{c}_2t$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings. If any such earring contains a green vertex, then it obviously contains at least two edges incident with a green vertex. Since these earrings are pairwise edge-disjoint, it immediately follows that \mathcal{E} has a subcollection \mathcal{E}' , with $|\mathcal{E}'| \geq |\mathcal{E}| - 2t$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings that do not contain any green vertex. That is, each earring in \mathcal{E}' is an (ℓ_0, Δ_0) -earring of G .

Therefore, \mathcal{E}' is a collection of at least $|\mathcal{E}| - 2t \geq \mathbf{c}_1|E(H)| - (\mathbf{c}_2 + 2)t$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings in G . Since $|E(H)| \geq |E|$, it follows that $|\mathcal{E}'| \geq \mathbf{c}_1|E| - (\mathbf{c}_2 + 2)t = \mathbf{c}_1|E| - \mathbf{c}_7t$. \square

5. THE EMBEDDING METHOD: ADDING EDGES WITH FEW CROSSINGS

Our main goal is to show that every (sufficiently large) irreducible graph has a large collection of edges whose removal leaves a graph with large crossing number. The first main ingredient is the existence of a large collection of pairwise edge-disjoint (ℓ, Δ) -earrings (for some fixed ℓ and Δ); this is Lemma 8. The second main ingredient is the *embedding method*, which was used under similar circumstances by Richter and Thomassen [22], Fox and

506 Tóth [10], and Černý, Kynčl and Tóth [5] (see also [13, 24, 26]). We use the
 507 embedding method to prove the following.

508 **Lemma 12.** *Let G be a graph, and let ℓ, Δ , and r be positive integers.*
 509 *Suppose that G has a collection of r pairwise edge-disjoint (ℓ, Δ) -earrings.*
 510 *Then G has a set E_0 of r edges such that $\text{cr}(G - E_0) > (1/2)\text{cr}(G) -$
 511 $(1/2)(\Delta\ell + \ell^2)r$.*

512 *Proof.* Let $\Xi_1, \Xi_2, \dots, \Xi_r$ be a collection of pairwise edge-disjoint (ℓ, Δ) -
 513 earrings in G . For $i = 1, 2, \dots, r$, let $e_i = u_i v_i$ be the base edge of Ξ_i ,
 514 and let P_i, Q_i be the $u_i v_i$ -paths such that $\Xi_i = P_i \cup Q_i \cup \{e_i\}$. We shall show
 515 that $E_0 := \{e_1, e_2, \dots, e_r\}$ satisfies the required property.

516 Let $t := \text{cr}(G - E_0)$, and let \mathcal{D} be a drawing of $G - E_0$ with t crossings.
 517 The strategy is to extend \mathcal{D} to a drawing of G by drawing e_i very close to
 518 either P_i or Q_i , for $i = 1, 2, \dots, r$. Our aim is to show that this can be done
 519 while adding relatively few crossings.

520 We analyze several types of crossings of P_i and Q_i , for $i = 1, 2, \dots, r$. A
 521 crossing in \mathcal{D} is (i) *of Type 1* if one edge is in P_i and the other edge is in
 522 Q_i , for some $i \in \{1, \dots, r\}$; (ii) *of Type 2A* if one edge is in $P_i \cup Q_i$ and
 523 the other edge is in $P_j \cup Q_j$, for some $i \neq j$, $i, j \in \{1, \dots, r\}$; and (iii) *of*
 524 *Type 2B* if one edge is in $P_i \cup Q_i$ for some $i \in \{1, \dots, r\}$ and the other
 525 in $E(G) \setminus \bigcup_{j=1}^r (P_j \cup Q_j)$. Note that if a crossing \times involving an edge of
 526 $\bigcup_{i=1}^r P_i \cup Q_i$ is neither of Type 1, nor 2A, nor 2B, then the edges involved
 527 in \times must be both in P_i or both in Q_i , for some $i \in \{1, 2, \dots, r\}$. As we shall
 528 see, this last type of crossing is irrelevant to our discussion.

529 For $i = 1, 2, \dots, r$ and $k \in \{1, 2\}$, let $\chi_k(P_i)$ (respectively, $\chi_k(Q_i)$) denote
 530 the number of crossings of Type k that involve an edge in P_i (respectively,
 531 Q_i).

532 In every crossing-minimal drawing of any graph, no pair of edges cross
 533 each other more than once. Since each of P_i and Q_i has at most ℓ edges, it
 534 follows that

$$(6) \quad \chi_1(P_i) \leq \ell^2, \text{ for } i = 1, \dots, r.$$

535 Now let \mathcal{R} be the set of all sequences (R_1, R_2, \dots, R_r) , with $R_i \in \{P_i, Q_i\}$
 536 for $i = 1, 2, \dots, r$, and consider the sum $\Sigma := \sum_{R \in \mathcal{R}} (\sum_{i=1}^r \chi_2(R_i))$.

537 We claim that a crossing of Type 2A contributes in exactly 2^r to Σ . To see
 538 this, first note that such a crossing involves an edge of an $R_i \in \{P_i, Q_i\}$ and
 539 an edge of an $R_j \in \{P_j, Q_j\}$ for some $i \neq j$. Let T_i (respectively, T_j) be the
 540 element in $\{P_i, Q_i\} \setminus R_i$ (respectively, $\{P_j, Q_j\} \setminus R_j$). There are 2^{r-2} sequences
 541 in \mathcal{R} that include both R_i and R_j , and so for each such sequence, the crossing
 542 contributes in 2 to Σ . There are 2^{r-2} sequences in \mathcal{R} that include R_i and do
 543 not include R_j , and so for each such sequence, the crossing contributes in 1
 544 to Σ . Analogously, there are 2^{r-2} sequences in \mathcal{R} that include R_j and do not
 545 include R_i , and so for each such sequence, the crossing contributes in 1 to Σ .
 546 Therefore each crossing of Type 2A contributes in $2 \cdot 2^{r-2} + 2^{r-2} + 2^{r-2} = 2^r$
 547 to Σ , as claimed. Note that this reasoning assumes that no crossing of Type

548 2A is in both P_i and Q_i for the same i . This is immediate if P_i and Q_i are
 549 edge-disjoint, but we recall from our definition of earring that P_i and Q_i may
 550 share edges. However, the validity of our reasoning follows since (again, by
 551 the definition of earring) any edge $f \in E(P_i) \cap E(Q_i)$ is a cut edge of $G - e_i$,
 552 from which it follows that f cannot be crossed in any optimal drawing of
 553 $G - E_0$.

554 We also note that a crossing of Type 2B contributes to Σ in exactly 2^{r-1} .
 555 Indeed, such a crossing involves (for some fixed i) an edge of R_i and an
 556 edge that belongs to no R_j ; it contributes in 1 to $\chi(R_i)$, and there are 2^{r-1}
 557 sequences in \mathcal{R} that include R_i . (As in the previous paragraph, we remark
 558 that we are making use of the valid assumption that no crossing is in both
 559 P_i and Q_i for the same i).

560 In conclusion, each crossing of Type 2A or 2B contributes to Σ in at
 561 most 2^r . Since only crossings of Types 2A and 2B contribute to Σ , and
 562 \mathcal{D} has t crossings in total, we conclude that $\sum_{R \in \mathcal{R}} (\sum_{i=1}^r \chi_2(R_i)) \leq 2^r t$.
 563 Since $|\mathcal{R}| = 2^r$, it follows that for some sequence $(R_1, R_2, \dots, R_r) \in \mathcal{R}$,
 564 $\sum_{i=1}^r \chi_2(R_i) \leq t$. By relabeling (exchanging) P_i and Q_i if necessary, we may
 565 assume without any loss of generality that $R_i = P_i$ for each $i = 1, 2, \dots, r$,
 566 and so

$$(7) \quad \sum_{i=1}^r \chi_2(P_i) \leq t.$$

567 Now note that some P_i may have self-crossings. However, for each i there
 568 is a simple curve α_i , contained in P_i , joining u_i and v_i . The definition
 569 of crossings of types 1, 2A, and 2B obviously extend to the crossings on
 570 each α_i , and so (6) and (7) imply that $\chi_1(\alpha_i) \leq \ell^2$ for $i = 1, 2, \dots, r$, and
 571 $\sum_{i=1}^r \chi_2(\alpha_i) \leq t$. Moreover (this is the effect of having obtained α_i by
 572 avoiding the self-crossings of its corresponding P_i), for $i = 1, 2, \dots, r$, each
 573 crossing of α_i is of one of these types.

574 The idea is to draw each e_i very close to its corresponding α_i . There are
 575 two kinds of crossings on the resulting drawings of e_i , $i = 1, \dots, r$. Some
 576 crossings occur as we traverse e_i and pass very close to a crossing of α_i .
 577 The inequalities in the previous paragraph imply that there are, in total, at
 578 most $\ell^2 r + t$ crossings of this first kind. The second kind of crossing occurs
 579 as we pass very close to a vertex in α_i , and cross some edges incident with
 580 this vertex. Since each such vertex is an internal vertex of some P_i (that is,
 581 has degree $< \Delta$) and there are at most $\ell - 1$ internal vertices in each P_i , we
 582 conclude that each e_i has fewer than $\Delta \ell$ crossings of this second kind. Thus
 583 in total there are fewer than $\Delta \ell r$ crossings of the second kind.

584 We conclude that all the edges e_1, e_2, \dots, e_r may be added to the drawing
 585 \mathcal{D} of $G - E_0$ by introducing fewer than $(\Delta \ell + \ell^2)r + t$ crossings. Since $t =$
 586 $\text{cr}(G - E_0)$, it follows that $\text{cr}(G) < 2\text{cr}(G - E_0) + (\Delta \ell + \ell^2)r$ or, equivalently,
 587 $\text{cr}(G - E_0) > (1/2)\text{cr}(G) - (1/2)(\Delta \ell + \ell^2)r$. \square

588 If we are interested in removing only one edge (as we are in Theorem 2),
 589 we can improve the $1/2$ coefficient in Lemma 12 to $2/3$, as the following
 590 statement shows.

591 **Lemma 13.** *Let G be a graph, and let ℓ and Δ be positive integers. Suppose*
 592 *that G has an (ℓ, Δ) -earring. Then G has an edge e such that $\text{cr}(G - e) >$*
 593 *$(2/3)\text{cr}(G) - (2/3)(\Delta\ell + \ell^2)$.*

594 *Proof.* The proof is essentially the same as the proof of Lemma 13, with
 595 the following favourable exception. If we consider only one earring, then
 596 $r = 1$, and so there are no crossings of Type 2A. Each crossing of Type 2B
 597 contributes to Σ in at most 1, and so $\chi_2(P_1) + \chi_2(Q_1) \leq t$. By exchanging
 598 P_1 and Q_1 if necessary, we may assume that $\chi_2(P_1) \leq t/2$.

599 In parallel to the last paragraph of the proof of Lemma 13, in the present
 600 case we conclude that the edge e_1 may be added to the drawing \mathcal{D} of $G - E_0 =$
 601 $G - e_1$ by introducing fewer than $(\Delta\ell + \ell^2) + t/2$ crossings. Since $t =$
 602 $\text{cr}(G - e_1)$, it follows that $\text{cr}(G) < (3/2)\text{cr}(G - e_1) + \Delta\ell + \ell^2$ or, equivalently,
 603 $\text{cr}(G - e_1) > (2/3)\text{cr}(G) - (2/3)(\Delta\ell + \ell^2)$. \square

604 6. PROOF OF THEOREMS 1 AND 2

605 *Proof of Theorem 1.* Let $\ell_0 := 5000$ and $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, and
 606 $\mathbf{c}_7 := (10^{-5} + 2)$. Let k be a positive integer and let $\epsilon > 0$. Define
 607 $\gamma := \epsilon / ((1/2)(\Delta_0\ell_0 + \ell_0^2))$ and $m_0 := ((\mathbf{c}_7 + \gamma)k) / \mathbf{c}_1$. Let $G = (V, E)$ be
 608 a 2-connected irreducible graph with $\text{cr}(G) = k$ and at least m_0 edges.

609 Lemma 11 implies that G has a collection of at least $\mathbf{c}_1|E| - \mathbf{c}_7k$ pairwise
 610 edge-disjoint (ℓ_0, Δ_0) -earrings. Since $|E| \geq ((\mathbf{c}_7 + \gamma)k) / \mathbf{c}_1$, it follows that
 611 G has a collection of at least γk pairwise edge-disjoint (ℓ_0, Δ_0) -earrings.
 612 Thus, by Lemma 12, G has a collection E_0 of at least γk edges such that
 613 $\text{cr}(G - E_0) > (1/2)\text{cr}(G) - (1/2)(\Delta_0\ell_0 + \ell_0^2)\gamma k = (1/2)\text{cr}(G) - \epsilon k = ((1/2) -$
 614 $\epsilon)\text{cr}(G)$. \square

615 If u, v are vertices of a graph G , a *double uv -path* is a subgraph of G that
 616 consists of a uv -path with all its edges doubled.

617 *Proof of Theorem 2.* Let $\ell_0 := 5000$, $\Delta_0 := 500$, $\mathbf{c}_1 := 10^{-10}$, and $\mathbf{c}_7 :=$
 618 $(10^{-5} + 2)$. Let k be a positive integer, and let $m_1 := (\mathbf{c}_7k) / \mathbf{c}_1 + 1$. We
 619 prove that if $G = (V, E)$ is a 2-connected graph in which each vertex is
 620 adjacent to at least 3 vertices, $\text{cr}(G) = k$, and G has at least m_1 edges, then
 621 G has an edge e such that $\text{cr}(G - e) > (2/3)\text{cr}(G) - 10^8$.

622 Suppose first that G is not irreducible, and let (B, u, v) be a minimal
 623 blob in G , (that is, G has no blob (B', u', v') such that B' is a subgraph of
 624 B). The minimality of B implies that B has no cut edges, and so its width
 625 $w(B)$ is at least 2. It is easy to see that if every edge of B is in a 2-edge-cut
 626 separating u and v , then B is a double uv -path. This clearly contradicts the
 627 X -minimality of G , and so we conclude that there is an edge e in B such
 628 that the uv -blob (in $G - e$) $B - e$ has width at least 2.

By way of contradiction, suppose that $\text{cr}(G - e) < (2/3)\text{cr}(G)$. It is straightforward to see that there is a crossing-minimal drawing \mathcal{D} of $G - e$ in which the set E' of edges crossed in $B - e$ form a smallest *uv-edge cut* (that is, a minimum size edge cut in $B - e$ separating u and v), with each edge in E' crossed the same number (say s) of times. In particular, $\text{cr}(G - e) \geq |E'|s \geq 2s$. The planarity of $B - e$ (with u, v in the same face) implies that: (i) if e is in distinct components of $(B - e) - E'$, then e can be added to \mathcal{D} by introducing exactly s crossings; and (ii) otherwise, e can be added to \mathcal{D} without introducing any crossings. In either case, the result is a drawing of G with at most $\text{cr}(G - e) + s$ crossings, and so $\text{cr}(G) \leq \text{cr}(G - e) + s$. The assumption $\text{cr}(G) > (3/2)\text{cr}(G - e)$ then implies $\text{cr}(G - e) < 2s$, contradicting that $\text{cr}(G - e) \geq 2s$. Thus $\text{cr}(G - e) \geq (2/3)\text{cr}(G) > (2/3)\text{cr}(G) - 10^8$.

Suppose finally that G is irreducible. Lemma 11 then implies that G has at least $\mathbf{c}_1|E| - \mathbf{c}_7k$ pairwise edge-disjoint (ℓ_0, Δ_0) -earrings. Since $|E| \geq (\mathbf{c}_7k)/\mathbf{c}_1 + 1$, it follows that G has at least one (ℓ_0, Δ_0) -earring. Thus, by Lemma 13, G has an edge e such that $\text{cr}(G - e) > (2/3)\text{cr}(G) - (2/3)(\Delta\ell + \ell^2) > (2/3)\text{cr}(G) - 10^8$. \square

7. BOUNDED DECAY AND EXPECTED CROSSING NUMBERS

The pioneering work of Richter and Thomassen, as well as our work in this paper, are naturally described as “bounded decay” results: the existence of sets of edges whose removal does not decrease arbitrarily the crossing number. The papers by Fox and Tóth [10] and by Černý, Kynčl and Tóth [5] concern themselves with “almost no decay” results: the existence of sets of edges whose removal results in a very small decrease of the crossing number.

As an additional motivation to bounded decay results, we discuss in this section a connection with expected crossing numbers, a concept recently introduced by Mohar and Tamon [18, 19].

7.1. Expected crossing numbers and decay of crossing numbers.

Given a drawing \mathcal{D} of a graph $G = (V, E)$, and a *weight function* $w : E \rightarrow \mathbb{R}_+$, define the *crossing weight* $\text{cr}(\mathcal{D}, w)$ as $\sum_{\{e, f\} \in \mathbb{X}(\mathcal{D})} w(e)w(f)$, where $\mathbb{X}(\mathcal{D})$ is the set of all pairs of edges that cross each other in \mathcal{D} . The pair (G, w) is a *weighted graph*, and the *weighted crossing number* of (G, w) is $\text{cr}(G, w) := \min_{\mathcal{D}} \text{cr}(\mathcal{D}, w)$, where the minimum is taken over all drawings \mathcal{D} of G . Now take the weights on the edges to be independently identically distributed random variables, with uniform distributions on the interval $[0, 1]$. The expected value of $\text{cr}(G, w)$ under this distribution is the *expected crossing number* of G , and is denoted $\mathbb{E}(\text{cr}(G))$.

Let us say that a family \mathcal{G} of graphs is *robust* (or, more precisely, *ϵ -robust*) if there exist a constant $\epsilon := \epsilon(\mathcal{G})$ and an $n(\mathcal{G})$ such that $\mathbb{E}(\text{cr}(G)) \geq \epsilon \cdot \text{cr}(G)$ for every graph G in \mathcal{G} with at least $n(\mathcal{G})$ vertices.

Mohar and Tamon proved in [18] that $\mathbb{E}(\text{cr}(K_n))$ is $\Theta(n^4)$. From this it follows immediately that the family of all complete graphs is robust. Moreover, it follows from their Crossing Lemma for Expectations (Theorem 5.2

in [18]) that for each fixed $\gamma > 0$, the family of graphs with at least $\gamma \cdot n^2$ edges is also robust (more precisely, ϵ -robust, where ϵ might depend on γ). It is thus natural to inquire about the robustness of families of sparser graphs.

Our aim in this subsection is to unveil and exploit the close connection between robustness and several results and conjectures, presented in [5], on the decay of crossing numbers.

In [5], Černý, Kynčl and Tóth proved the following: for each $\epsilon > 0$, there exist $\delta, \gamma > 0$ such that every sufficiently large graph G with n vertices and $m \geq n^{1+\epsilon}$ edges has a subgraph G' with at most $(1 - \delta)m$ edges such that $\text{cr}(G') \geq \gamma \cdot \text{cr}(G)$. This impressive “almost no decay” statement is best possible, in the sense that (as shown in [5]) one cannot require that *every* subgraph with $(1 - \delta)m$ edges has crossing number at least $\gamma \cdot \text{cr}(G)$. In this vein, Černý, Kynčl and Tóth also investigated the following closely related problem.

Let us say that a family \mathcal{G} of graphs is *stable* (or, more precisely, (δ, γ) -stable) if there exist positive constants $\delta := \delta(\mathcal{G})$, $\gamma := \gamma(\mathcal{G})$, and $n(\mathcal{G})$ such that for every graph $G \in \mathcal{G}$ with at least $n(\mathcal{G})$ vertices (and m edges), a positive fraction of all subgraphs of G with $(1 - \delta)m$ edges has crossing number at least $\gamma \cdot \text{cr}(G)$. The requirement may be equivalently formulated as follows: if G' is a random subgraph of G obtained by deleting independently each edge with probability δ , then w.h.p. $\text{cr}(G') \geq \gamma \cdot \text{cr}(G)$.

In the earlier version [4] of [5], it was conjectured that for each $\epsilon > 0$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is stable. In [5], it was shown that this is false for $\epsilon < 1/3$ (we have slightly refined the construction in [5], and shown that it does not hold either for $\epsilon = 1/3$; see Theorem 17). The conjecture remains open for denser graphs:

Conjecture 14. *There exists an $\bar{\epsilon} \in (1/3, 1)$ such that, for each $\epsilon \in (\bar{\epsilon}, 1]$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is stable.*

(See also a weaker version put forward in [5]).

Before moving on to explore the close relationship between Conjecture 14 and the robustness of dense graphs, we note the stability of random graphs:

Remark 15. *The family of all random graphs $G(n, p)$ with $p > 2/n$, is stable.*

Proof. We start by noting that $\mathbb{E}(\text{cr}(G(n, p))) \leq p^2 \text{cr}(K_n) \leq (1/10)p^2 n^4$. From the other side, Spencer and G. Tóth ([25], Section 4) proved that there is a $c > 0$ such that for n sufficiently large the lower bound $\mathbb{E}(\text{cr}(G(n, 2/n))) > cn^2$ holds. Standard sparsening of $G(n, p)$ (keeping each edge with probability $2/(pn)$) gives that for $p > 2/n$, $\mathbb{E}(\text{cr}(G(n, p))) > (c/4)p^2 n^4$. Using these bounds, together with the observation that if each edge of a $G(n, p)$ is removed with probability ϵ then we obtain a $G(n, (1 - \epsilon)p)$, the remark follows. \square

The key connection between expected crossing number (robustness) and the decay of crossing numbers (stability) is the following observation:

Proposition 16. *If a family \mathcal{G} of graphs is stable, then it is robust. More precisely: if \mathcal{G} is (δ, γ) -stable, then it is $\delta^2\gamma$ -robust.*

Proof. Suppose that \mathcal{G} is a (δ, γ) -stable family of graphs. Let G be a (sufficiently large) graph in \mathcal{G} , and let w be a random weight assignment (sampled from the uniform distribution) on the edges of G . Our aim is to show that the expected value of $\text{cr}(G, w)$ is at least $\delta^2\gamma \cdot \text{cr}(G)$.

Let G' be the subgraph of G that results by deleting the edges that receive a weight smaller than δ under w . Let \mathcal{D} be a drawing of G that minimizes $\text{cr}(G, w)$, and let \mathcal{D}' be the restriction of G to G' . Clearly \mathcal{D}' has at most $\text{cr}(G, w)/\delta^2$ crossings, and so $\text{cr}(G') \leq \text{cr}(\mathcal{D}') \leq \text{cr}(G, w)/\delta^2$. Thus $\text{cr}(G, w) \geq \delta^2 \text{cr}(G')$.

Note that G' may be equivalently regarded as a graph obtained from G by deleting each edge independently with probability δ . Since \mathcal{G} is (δ, γ) -stable, it follows that w.h.p. $\text{cr}(G') \geq \gamma \cdot \text{cr}(G)$. Therefore the expected value of $\text{cr}(G, w)$ is at least $\delta^2\gamma \cdot \text{cr}(G)$, as required. \square

We now proceed with a concrete illustration of how the results and techniques on the decay of crossing numbers (specifically, those developed in [5]) find an immediate application in expected crossing numbers.

As we observed above, Černý, Kynčl and Tóth [5] proved that, for each $\epsilon \in (0, 1/3)$, the family of graphs with $\Theta(n^{1+\epsilon})$ edges is *not* stable. We have slightly refined the construction in [5], and extended it to cover the case $\epsilon = 1/3$.

Theorem 17 (Non-stability of graphs with $\Theta(n^{4/3})$ edges). *For every $\delta, \gamma > 0$ there exist $c := c(\delta, \gamma)$ and $n_0 := n_0(\delta, \gamma)$ such that there exist infinitely many graphs G with $n > n_0$ vertices and $c \cdot n^{4/3} < m < n^{4/3}$ edges, that satisfy the following. If G' is a random subgraph of G obtained by deleting independently each edge with probability δ , then w.h.p.*

$$\text{cr}(G') < \gamma \cdot \text{cr}(G).$$

We omit the proof of this result, since it closely resembles the proof of our next statement. Theorem 18 shows the non-robustness of graphs with $\Theta(n^{4/3})$ edges, and illustrates how the non-stability results and techniques in [5] can be extended to prove the non-robustness of graphs with $\Theta(n^{1+\epsilon})$ edges for each $\epsilon \in (0, 1/3)$.

Theorem 18 (Non-robustness of graphs with $\Theta(n^{4/3})$ edges). *For every $\gamma > 0$ there exist $c := c(\gamma)$ and $n_0 := n_0(\gamma)$ such that there are infinitely many graphs G with $n > n_0$ vertices and $c \cdot n^{4/3} < m < n^{4/3}$ edges, and*

$$\mathbb{E}(\text{cr}(G)) < \gamma \cdot \text{cr}(G).$$

Proof. For readability purposes, we shall omit explicitly taking the integer part of several quantities involved. The integrality requirement will be, in every case, obvious from the context.

745 We may assume without loss of generality that γ is small enough so that
 746 $e^{-1200/\gamma} < \gamma/720$. Let $\alpha := \gamma/600$, $c := \alpha^2/100$, $r := \alpha^2 n^{1/3}/5$, $s := 1/\alpha^2$,
 747 and $t := \sqrt{n/s}$. Note that obviously $r > 5cn^{1/3}$.

748 Inspired by the construction in [5], G will be the disjoint union of two
 749 graphs G_1 and G_2 plus some isolated vertices. Let G_1 be $n/2r$ copies of the
 750 complete graph K_r . Clearly $|V(G_1)| \leq n/2$. Now let G_2 be obtained from a
 751 complete graph K_t by subdividing each edge $s-1$ times, i.e. replacing each
 752 edge by a path with s edges (these length s paths are the *branches*). It is
 753 easy to check that $|V(G_2)| \leq n/2$. Furthermore,

$$(8) \quad \alpha^4 n^2 = t^4 > \text{cr}(G_2) > \frac{t^4}{100} = \frac{n^2}{100s^2} = \frac{\alpha^4 n^2}{100},$$

754 where the inequalities $t^4 > \text{cr}(G_2) > t^4/100$ are easily derived bounds for
 755 the crossing number of the complete graph on t vertices.

756 Now let w be a random weight assignment on the edges of G . Let $E_{<\alpha}$
 757 denote the set of edges of G that receive a weight smaller than α under
 758 w . Let us say that a branch is *weak* if at least one of its edges is in $E_{<\alpha}$;
 759 otherwise the branch is *strong*.

The probability that any fixed branch is strong is

$$(1 - \alpha)^s \approx e^{-\alpha s} = e^{-1/\alpha}.$$

760 Using Chernoff's bound, w.h.p. at most $t^2 e^{-1/\alpha}$ branches are strong. That
 761 is, w.h.p. at least $\binom{t}{2} - t^2 e^{-1/\alpha} \approx t^2(1/2 - e^{-1/\alpha})$ branches are weak.

762 Now consider the drawing of G_2 in which the t vertices of degree $t-1$ are
 763 in convex position, and the edges are the straight segments joining them.
 764 This drawing of G_2 has $\binom{t}{4} \approx t^4/24$ crossings (this is by no means a crossing-
 765 minimal drawing of G_2 , but it is enough for our purposes). Moreover, by
 766 adjusting the drawing of each branch if needed, we may ensure that each
 767 branch is crossed in exactly one edge, namely the edge with smallest weight.
 768 It follows that the number of crossings involving two strong branches (and
 769 thus, in particular, the number of crossings of weight $\geq \alpha$) is w.h.p. at most
 770 $(t^2 e^{-1/\alpha})^2$, and so w.h.p.

$$(9) \quad \begin{aligned} \text{cr}(G_2, w) &< t^4 e^{-2/\alpha} + \alpha \cdot t^4(1/24 - e^{-2/\alpha}) < t^4(\alpha/24 + e^{-2/\alpha}) \\ &< 100 \text{cr}(G_2)(\alpha/24 + e^{-2/\alpha}) \leq 5\alpha \cdot \text{cr}(G_2), \end{aligned}$$

771 where for this last inequality we used that $e^{-1200/\gamma} = e^{-2/\alpha} < \gamma/720 =$
 772 $(5/6)\alpha$.

We finally move on to G . First we note that

$$|E(G)| = |E(G_1)| + |E(G_2)| \geq |E(G_2)| = (n/2r)r(r-1)/2 > nr/5 > cn^{4/3}.$$

773 Using (8), we obtain

$$(10) \quad \text{cr}(G) = \text{cr}(G_1) + \text{cr}(G_2) > \text{cr}(G_2) > \alpha^4 n^2 / 100.$$

From the other side, using (8) and (9) and the trivial bound $\text{cr}(K_r) \leq r^4$, we get

$$(11) \quad \text{cr}(G, w) \leq \text{cr}(G_1) + \text{cr}(G_2, w) \leq (n/2r)r^4 + 5\alpha^5 n^2 < 6\alpha^5 n^2,$$

where for the last inequality we used the (easily checked) inequality $(n/2r)r^4 < \alpha^5 n^2$.

Finally, using (10) and (11) and recalling that $\alpha = \gamma/600$, we obtain

$$\text{cr}(G, w) < 6\alpha^5 n^2 = (600\alpha)(\alpha^4 n^2/100) < \gamma \cdot \text{cr}(G),$$

as required. \square

We close this subsection with two constructions that further illustrate the discrepancy between the crossing number of a graph and its expected crossing number.

First we describe a construction that highlights the fact that the crossing number (of a family of graphs) may grow with the number of vertices, and yet the expected crossing number (of all graphs in the family) may be bounded by an absolute constant. For any graph G , let $n(G)$ and $m(G)$ denote the number of vertices and edges of G , respectively, and let $s \cdot G$ the graph that consists of s disjoint copies of G . Let $K_5(t)$ denote the graph obtained by replacing each edge of K_5 with a path of length t (a *branch*). Trivially, for any positive integer s , $n(s \cdot K_5(t)) = s(10(t-1) + 5) = 10st - 5s$, $m(s \cdot K_5(t)) = 10st$, and $\text{cr}(s \cdot K_5(t)) = s$. However, the weighted crossing number of $K_5(t)$ is $\min w(e)w(f)$, where the minimum is taken over all pairs of edges e, f that lie on branches that correspond to nonincident edges. A fairly standard calculation shows that $\mathbb{E}(\text{cr}(s \cdot K_5(t))) \leq (s/t^2) \log^2 s$. It is worthwhile to explore the consequences of plugging in various values of s . Probably the most interesting case occurs when $s = n^{2/3}/\log n$, for this shows the following:

Proposition 19. *There exists an infinite family of graphs G with crossing number $n^{2/3}/\log n$ and expected crossing number at most 1.* \square

Our final construction pertains a family of graphs that seem more natural than the graphs constructed above. We recall that $C_3 \square C_n$ denotes the Cartesian product of the cycles of sizes 3 and n (see Figure 1).

Proposition 20. *The Cartesian products $C_3 \square C_n$ satisfy*

$$\text{cr}(C_3 \square C_n) = n,$$

and yet

$$\mathbb{E}(\text{cr}(C_3 \square C_n)) \leq 2n^{2/3} \log^{1/3} n + 3.$$

Proof. The vertices of $C_3 \square C_n$ can be labeled $v_{i,j}$, $0 \leq i \leq 2$, $0 \leq j \leq n-1$, so that there is an edge joining $v_{i,j}$ and $v_{i',j'}$ if and only if either (i) $j = j'$ and $|i - i'| = 1$ or (ii) $i = i'$ and $|j - j'| = 1$ (indices are modulo n). For $j = 0, 1, \dots, n-1$, let $V_j := \{v_{i,j} \mid i \in \{0, 1, 2\}\}$. That is, the V_j s are the vertex sets of the 3-cycles. For $j = 0, 1, \dots, n-1$, let $E(j)$ denote the set of (three) edges with an endpoint in V_j and another endpoint in V_{j+1} .

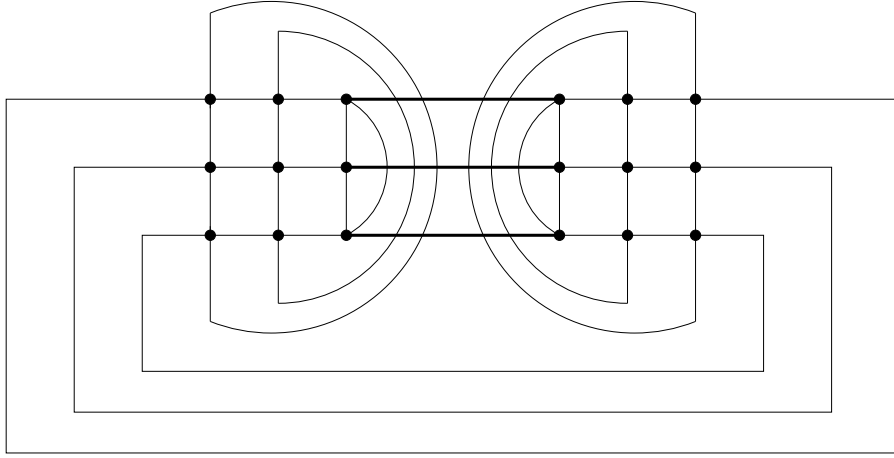


FIGURE 1. A drawing of $C_3 \square C_6$ with 14 crossings, where the thick edges are the edges of one particular $E(j)$. This is easily generalized to obtain, for every even integer $n \geq 2$, a (not crossing-minimal) drawing of $C_3 \square C_n$ with $3n - 4$ crossings with the following property: there exists a $j \in \{0, 1, 2, \dots, n-1\}$ such that each crossing involves an edge in $E(j)$.

808

809 It is known that $\text{cr}(C_3 \square C_n) = n$ for every $n \geq 3$ [23]. In Figure 1 we
 810 depict how to produce a (not crossing-minimal) drawing of $C_3 \square C_n$ with
 811 $3n - 4$ crossings, for every even integer $n \geq 2$, with the following property:
 812 there is a $j \in \{0, 1, 2, \dots, n-1\}$ such that every crossing involves an edge in
 813 $E(j)$ (the edges in $E(j)$ are the thick edges in Figure 1). Thus,

- 814 (A) if the edges in $C_3 \square C_n$ are weighted, and there exists a j such
 815 that the sum of the weights of the edges in $E(j)$ is r , then such a
 816 weighted $C_3 \square C_n$ has crossing number at most $r \cdot n$.

For $j = 0, 1, \dots, n-1$, denote the weights of the edges in $E(j)$ by x_1^j, x_2^j, x_3^j . We have for $t \leq 1$ that $\Pr(x_1^j + x_2^j + x_3^j > t) = 1 - t^3/3!$. Using independence,

$$\Pr(\exists j : x_1^j + x_2^j + x_3^j \leq t) = 1 - (1 - t^3/6)^n \approx 1 - \exp[-nt^3/6].$$

817 Choosing $t = 6^{1/3}n^{-1/3} \log^{1/3} n$, this is at least $1 - 1/n$.

818 Now let $s := \min\{x_1^j + x_2^j + x_3^j \mid j \in \{0, 1, \dots, n-1\}\}$. Thus $s \leq t$ with prob-
 819 ability at least $1 - 1/n$. In the complementary scenario (which occurs with
 820 probability $< 1/n$), s is obviously at most 3. Using this observation together
 821 with (A), it follows that $\mathbb{E}(\text{cr}(C_3 \square C_n)) < [(1 - 1/n)((6)^{1/3}n^{-1/3} \log^{1/3} n) +$
 822 $(1/n)3] \cdot n < 2n^{2/3} \log^{1/3} n + 3$. \square

823 **7.2. Concentration of the expected crossing number and the cross-**
 824 **ing number of randomly sparsened graphs.** Continuing in the theme
 825 of expected crossing numbers and its interplay with the decay of crossing
 826 numbers, we finally explore the concentration around the crossing number
 827 of a randomly sparsened graph, as well as the concentration around the
 828 expected crossing number of a graph.

829 Denote $R = R(G, p)$ the random graph obtained from G by randomly
 830 and independently removing edges, each with probability p . Using a stan-
 831 dard martingale concentration inequality we show that $\text{cr}(R)$ is concentrated
 832 around its mean. Let $E(G) = \{e_1, \dots, e_m\}$, and consider the random vari-
 833 able $\text{cr}(R)$ as a Doob's martingale, where the edges are exposed one by one.
 834 The length of the martingale is $|E(G)|$. Removing or adding an edge changes
 835 the crossing number by at most $|E(G)|$. Thus, by the Azuma-Hoeffding's
 836 inequality, for every $\lambda > 0$ we have

$$(12) \quad \Pr[|\mathbb{E}(\text{cr}(R)) - \text{cr}(R)| > \lambda] \leq \exp\left[\frac{-\lambda^2}{2|E(G)|^3}\right].$$

837 Let $\beta(n)$ be any function tending to infinity. Inequality (12) shows con-
 838 centration with radius $\lambda = \beta(n)|E(G)|^{3/2}$:

$$(13) \quad \Pr[|\mathbb{E}(\text{cr}(R)) - \text{cr}(R)| > \beta(n)|E(G)|^{3/2}] \leq \exp\left[\frac{-\beta(n)^2}{2}\right].$$

839 Similary, we can get concentration around the expected crossing number.
 840 Assign to each edge a random variable taking values from $[0, 1]$ (which could
 841 be different for each edge), which provides to each of them a random weight.
 842 Formally, it could be a function $w : E(G) \rightarrow \mathcal{F}$, where \mathcal{F} is a collection of
 843 random variables taking values from $[0, 1]$. Then $\mathbb{E}(\text{cr}(G, w))$ is the expected
 844 crossing number for a given w , and $\text{cr}(G, w)$ is a random variable, which is
 845 the crossing number of a weighted graph G . As with the random graph
 846 R above, resampling the weight of one edge changes the weighted crossing
 847 number by at most $|E(G)|$, and so we obtain:

$$(14) \quad \Pr[|\mathbb{E}(\text{cr}(G)) - \text{cr}(G, w)| > \beta(n)|E(G)|^{3/2}] \leq \exp\left[\frac{-\beta(n)^2}{2}\right].$$

848 These inequalities are meaningful only when G is dense enough, i.e. $|E(G)| \geq$
 849 $n^{5/4}$. Note that we could have obtained sharper concentration results for
 850 sparse graphs, under the assumption that removing any edge makes the
 851 crossing number drop by $o(|E(G)|)$.

852 8. CONCLUDING REMARKS

853 Lemma 8 falls into the realm of light subgraphs. We recall that the
 854 *weight* of a subgraph H of a graph G is the sum of the degrees (in G) of its
 855 vertices. For a class \mathcal{G} of graphs, define $w(H, \mathcal{G})$ as the smallest integer w

such that each graph $G \in \mathcal{G}$ which contains a subgraph isomorphic to H has a subgraph isomorphic to H of weight at most w . If $w(H, \mathcal{G})$ is finite then H is *light* in \mathcal{G} .

Fabricsi and Jendrol' [8] proved that paths (and no other connected graphs) are light in the class of 3-connected planar graphs. Fabricsi et al. [9] proved that this remains true even if the minimum degree is at least 4, and Mohar [16] extended this to 4-connected planar graphs.

Although some cycles are light in certain families of planar graphs (see for instance [11, 12, 15, 17]), it is easy to see that cycles are not light on the class of planar graphs (consider, for instance, a wheel W_n with n large: each cycle in W_n is either very long or incident with a large degree vertex). However, as Richter and Thomassen illustrated in [22], for some applications one does not need the full lightness condition. A cycle C in a graph is (ℓ, Δ) -*nearly light* if it has length less than ℓ and at most one of its vertices has degree Δ or greater. Richter and Thomassen proved that every planar graph has a $(6, 11)$ -nearly light cycle. This was later refined in [14], where it was shown that if the graphs under consideration are sufficiently large, then there is a $\Delta > 0$ such that a linear proportion of the face boundaries are $(6, \Delta)$ -nearly light.

The concept of (ℓ, Δ) -earrings extends the idea of nearly light cycles: we allow both vertices u, v incident with some edge e to have arbitrarily large degree, and ask for the existence of *two* cycles that contain e , have bounded length, and (other than u and v) bounded degree. The following immediate corollary (since every 3-connected graph is obviously irreducible) of Lemma 11 guarantees the existence of many pairwise edge-disjoint earrings in 3-connected planar graphs.

Lemma 21. *If $G = (V, E)$ is a 3-connected planar graph, then G has at least $10^{-10}|E|$ pairwise edge-disjoint $(5000, 500)$ -earrings.*

We remark that the linear dependence on $|E|$ in Lemma 21 is clearly best possible, since there cannot be more pairwise edge-disjoint earrings than edges in a graph.

Finally, it is natural to ask if the 3-connectedness requirement can be weakened. The construction illustrated in Figure 2 answers this in the negative.

It might be argued that the graphs constructed in the proof of Theorem 18 are somewhat artificial, since many edges are subdivided a large number of times. However, these graphs can be turned into 3-connected graphs, with equivalent properties, as follows. Consider the graph G_2 in the proof of Theorem 18, and some fixed drawing of G_2 (for instance, as in the proof of Theorem 18, draw the degree $t - 1$ vertices on a circumference, and the branches as the straight edges joining them). Let u_1, u_2, \dots, u_t be the *nodes* (degree $t - 1$ vertices) of G_2 . Thus each branch with endpoints u_i, u_j can be written as $u_i = u_{i,j}^0, u_{i,j}^1, \dots, u_{i,j}^{s-1}, u_{i,j}^s = u_j$ (the same branch, traversing the

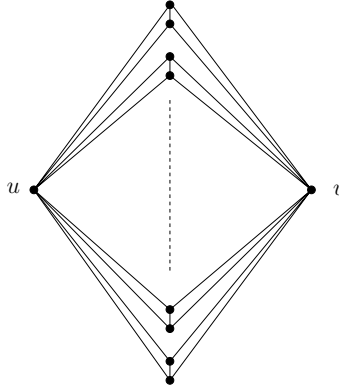


FIGURE 2. The graph H_n obtained by identifying n copies of $K_4 - e$ on their degree 2 vertices u, v . This family of 2-connected graphs shows that the 3-connectedness condition in Lemma 21 cannot be weakened: for each pair of integers ℓ, Δ there is an $n_0 := n_0(\ell, \Delta)$ such that for all $n \geq n_0$, H_n does not contain any (ℓ, Δ) -earring.

900 vertices in the reverse order, reads $u_j = u_{j,i}^0, u_{j,i}^1, \dots, u_{j,i}^{s-1}, u_{j,i}^s = u_i$, so that
 901 $u_{i,j}^k = u_{j,i}^{s-k}$ for $k = 0, 1, \dots, s$). Now for each branch $u_{i,j}^0, u_{i,j}^1, \dots, u_{i,j}^{s-1}, u_{i,j}^s$,
 902 add the edges $u_{i,j}^k$ and $u_{i,j}^{k+2}$, for $k = 0, 1, \dots, s-2$. The augmented graph is
 903 already 2-connected, but each pair of nodes (that is, degree $t-1$ vertices)
 904 is a 2-vertex-cut, so we need to strengthen the connectivity around each
 905 node. Consider the node u_1 , and suppose for simplicity that the edges
 906 $u_1 u_{1,2}^1, u_1 u_{1,3}^1, \dots, u_1 u_{1,t}^1$ leave u_1 in the given (say clockwise) cyclic order.
 907 Then, for each $j = 2, 3, \dots, s$, it is possible to draw an edge from one of
 908 $u_{1,j}^1$ and $u_{1,j}^2$ to one of $u_{1,j+1}^1$ and $u_{1,j+1}^2$ without introducing any crossings
 909 (indices are read modulo s). By performing this procedure around each
 910 node, we obtain a 3-connected graph that also witnesses Theorem 18. The
 911 proof is analogous to the proof of Theorem 18; the only difference is that
 912 instead of requiring a weak edge of a branch (say between u_i and u_j), we
 913 need weak triplets of edges of the form $(u_{i,j}^\ell, u_{i,j}^{\ell+1}), (u_{i,j}^{\ell-1}, u_{i,j}^{\ell+1}), (u_{i,j}^\ell, u_{i,j}^{\ell+2})$,
 914 where $3 \leq \ell \leq s-3$; we omit the details.

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